

Optimal Generator Design for Quantum Coherent Lyapunov Feedback

*Full Expressivity of the Ring-Topology $\sigma_X^{(k)}\sigma_Z^{(k\oplus 1)}$ Generator Set and Its
Implications for Measurement-Efficient Quantum Machine Learning*

Mark Luo

SuperQubit Inc., Markham, ON, Canada
mluo@superqubit.ai

June 2026

Abstract

We study the algebraic structure of parameterized quantum circuits (PQCs) generated by the ring-topology nearest-neighbour entangling operators

$$G_k = \sigma_X^{(k)} \otimes \sigma_Z^{(k\oplus 1)}, \quad k = 1, \dots, n, \quad k \oplus 1 := (k \bmod n) + 1,$$

where the last qubit couples back to the first, forming a closed ring. Three main results are established. **(1) Algebraic foundations:** We introduce the Dynamical Lie Algebra (DLA) of a PQC, prove the reachability theorem linking the DLA to circuit expressivity, and state the Baker–Campbell–Hausdorff (BCH) theorem with its role in connecting Lie algebras to Lie groups. **(2) Full expressivity for the ring generator set:** We prove in detail that the ring generator set satisfies $\mathfrak{g} = \mathfrak{su}(2^n)$ for all $n \geq 2$. The ring topology adds the wrap-around generator $G_n = \sigma_X^{(n)}\sigma_Z^{(1)}$ absent in the open chain, which contributes new cross-boundary commutators that accelerate closure and reduce the commutator depth needed for full spanning. **(3) CLF optimality:** We show that the ring generator set simultaneously optimises all four CLF performance criteria: global convergence (no spurious local maxima), hardware efficiency (nearest-neighbour two-qubit gates only), measurement reduction (the ring symmetry halves the number of independent gradient components via a discrete translational symmetry), and barren-plateau resistance. In particular, the ring symmetry implies $\partial F/\partial\theta_k = \partial F/\partial\theta_l$ at symmetric parameter configurations, reducing the number of independent COM measurements from M to M/n , a factor-of- n reduction in measurement overhead compared to the open chain.

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1 Introduction and Overview

Parameterized quantum circuits are the workhorse of near-term quantum machine learning (QML). A PQC is a unitary

$$U(\boldsymbol{\theta}) = \prod_{k=1}^M e^{-i\theta_k G_k}, \quad \boldsymbol{\theta} = (\theta_1, \dots, \theta_M) \in \mathbb{R}^M,$$

where the Hermitian *generators* $\{G_k\}$ are fixed by the circuit architecture and the parameters $\boldsymbol{\theta}$ are trained. The training objective in state-preparation QML is to maximise the fidelity

$$F(\boldsymbol{\theta}) = |\langle \psi_{\text{target}} | U(\boldsymbol{\theta}) | \psi_0 \rangle|^2$$

between the circuit output and a target state $|\psi_{\text{target}}\rangle$.

Two central questions in PQC design are:

1. *Expressivity*: Can $U(\boldsymbol{\theta})$ approximate any target unitary, or is it restricted to a submanifold?
2. *Trainability*: Does the fidelity landscape permit efficient optimization, or are there barren plateaus and spurious local optima?

Both questions are answered by the *Dynamical Lie Algebra* of the generator set.

1.1 The CLF Training Algorithm

The Coherent Lyapunov Feedback (CLF) algorithm [1] trains PQCs by a continuous feedback law derived from Lyapunov stability theory, eliminating the $\mathcal{O}(M)$ shot overhead of the parameter-shift rule. The CLF update law is:

$$\frac{d\theta_k}{dt} = \eta_k \frac{\partial F}{\partial \theta_k} = 2\eta_k \text{Im}[A_k(\boldsymbol{\theta}) A(\boldsymbol{\theta})^*],$$

where $A(\boldsymbol{\theta}) = \langle \psi_{\text{target}} | U(\boldsymbol{\theta}) | \psi_0 \rangle$ and $A_k(\boldsymbol{\theta}) = \langle \psi_{\text{target}} | G_k U(\boldsymbol{\theta}) | \psi_0 \rangle$ are complex overlap amplitudes extracted from the Coherent Overlap Module (COM) — a two-register quantum circuit that computes these amplitudes without projective measurement of the main register. The Lyapunov function $V = 1 - F$ decreases monotonically under this law.

Global convergence of CLF requires the DLA to equal $\mathfrak{su}(2^n)$. The generator choice $G_k = \sigma_X^{(k)} \sigma_Z^{(k\oplus 1)}$ on a *ring* topology is shown in this paper to achieve this while providing additional performance benefits through its discrete translational symmetry.

1.2 The Ring vs. the Open Chain

The open chain (studied in [2]) uses generators $G_k = \sigma_X^{(k)} \sigma_Z^{(k+1)}$ for $k = 1, \dots, n-1$. The ring adds the wrap-around generator:

$$G_n = \sigma_X^{(n)} \otimes \sigma_Z^{(1)},$$

closing the chain into a loop. This single additional generator has dramatic consequences:

- It creates cross-boundary commutators $[G_{n-1}, G_n]$ and $[G_n, G_1]$ that generate new operators spanning sites $n, 1, 2$ and $n - 1, n, 1$ respectively.
- The ring has a \mathbb{Z}_n discrete translational symmetry $k \mapsto k + 1 \pmod{n}$, which implies G_{k+1} is the cyclic translate of G_k . This symmetry is inherited by the fidelity landscape and halves the number of independent gradient measurements.
- Commutator depth to full closure is reduced: the chain requires depth $\mathcal{O}(n)$ while the ring achieves it in depth $\mathcal{O}(n/2)$ by using both “clockwise” and “counterclockwise” commutator paths.

1.3 Paper Organisation

Section 2 introduces DLAs and proves the reachability theorem. Section 3 develops the BCH theorem. Section 4 establishes Pauli algebra foundations. Section 5 contains the main proof of $\mathfrak{g} = \mathfrak{su}(2^n)$ for the ring generator set. Section 6 analyses the four CLF benefits in detail.

2 Dynamical Lie Algebras

2.1 Lie Algebras

Definition: Lie Algebra

A *Lie algebra* $(\mathfrak{g}, [\cdot, \cdot])$ is a real or complex vector space \mathfrak{g} with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying:

- (i) **Antisymmetry:** $[X, Y] = -[Y, X]$.
- (ii) **Jacobi identity:** $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

The map $[\cdot, \cdot]$ is the *Lie bracket*. For matrix algebras, $[X, Y] = XY - YX$.

The fundamental Lie algebra of n -qubit quantum computing is:

$$\mathfrak{su}(2^n) = \{A \in M_{2^n}(\mathbb{C}) : A^\dagger = -A, \text{Tr}(A) = 0\},$$

with $\dim_{\mathbb{R}} \mathfrak{su}(2^n) = 4^n - 1$. Working with Hermitian generators G_k (rather than skew-Hermitian iG_k), we identify $\mathfrak{su}(2^n) \cong i \cdot \{\text{traceless Hermitian matrices}\}$.

2.2 The Dynamical Lie Algebra

Definition: Dynamical Lie Algebra

For generators $\{G_k\}_{k=1}^M$ on $(\mathbb{C}^2)^{\otimes n}$, the *Dynamical Lie Algebra* is the smallest real Lie algebra containing $\{G_k\}$:

$$\begin{aligned} \mathfrak{g} &= \text{Lie}\{G_1, \dots, G_M\} \\ &= \text{span}_{\mathbb{R}}\{G_k, [G_j, G_k], [[G_j, G_k], G_l], [[[G_j, G_k], G_l], G_m], \dots\}. \end{aligned}$$

The closure process terminates in at most $4^n - 1$ steps since $\dim \mathfrak{su}(2^n) < \infty$.

Theorem: Reachability

Let $\mathcal{G} = \exp(\mathfrak{g}) = \{e^X : X \in \mathfrak{g}\}$ be the Lie group of the DLA. The set of unitaries reachable by $U(\boldsymbol{\theta}) = \prod_k e^{-i\theta_k G_k}$ over all $\boldsymbol{\theta} \in \mathbb{R}^M$ is exactly \mathcal{G} . In particular:

$$\mathfrak{g} = \mathfrak{su}(2^n) \iff \mathcal{G} = SU(2^n) \iff \text{PQC is maximally expressive.}$$

Proof. Products $\prod_k e^{-i\theta_k G_k}$ generate a connected Lie subgroup of $SU(2^n)$ with Lie algebra equal to \mathfrak{g} (Lie correspondence theorem). Since $SU(2^n)$ is compact and connected, a subgroup with algebra $\mathfrak{g} = \mathfrak{su}(2^n)$ must equal $SU(2^n)$. \square

Theorem: No Spurious Local Maxima

If $\mathfrak{g} = \mathfrak{su}(2^n)$, then $F(\boldsymbol{\theta})$ has no local maxima with $F < 1$. Every sub-optimal critical point is a saddle point.

Proof. With $\mathfrak{g} = \mathfrak{su}(2^n)$, the gradient $\nabla_{\boldsymbol{\theta}} F$ vanishes at $\boldsymbol{\theta}^*$ only if $G_k U(\boldsymbol{\theta}^*) |\psi_0\rangle$ is orthogonal to $|\psi_{\text{target}}\rangle$ for all k . Since $\{G_k\}$ generates $\mathfrak{su}(2^n)$, these conditions together force $U(\boldsymbol{\theta}^*)$ to lie in the stabiliser of $|\psi_{\text{target}}\rangle$ up to orthogonal complement, making any such critical point unstable under perturbation by elements of $\mathfrak{su}(2^n) \setminus \text{Stab}(U(\boldsymbol{\theta}^*) |\psi_0\rangle)$. \square

2.3 Lie Algebra Dimensions and Inclusions

The chain of inclusions relevant to our generator sets:

$$\bigoplus_{k=1}^n \mathfrak{su}(2) \subsetneq \mathfrak{su}(2)^{\otimes n} \subsetneq \mathfrak{su}(4) \subsetneq \dots \subsetneq \mathfrak{su}(2^n),$$

where:

Generator type	DLA	Dimension
Single-site only: $\sigma_X^{(k)}$	$\bigoplus_k \mathfrak{su}(2)$	$3n$
Open chain: $\sigma_X^{(k)} \sigma_Z^{(k+1)}$, $k = 1 \dots n - 1$	$\mathfrak{su}(2^n)$	$4^n - 1$
Ring: $\sigma_X^{(k)} \sigma_Z^{(k \oplus 1)}$, $k = 1 \dots n$	$\mathfrak{su}(2^n)$	$4^n - 1$

The ring and chain have the same DLA, but the ring achieves it with greater efficiency due to its \mathbb{Z}_n symmetry.

3 The Baker–Campbell–Hausdorff Theorem

3.1 Statement

Theorem: Baker–Campbell–Hausdorff (BCH)

Let X, Y be elements of a matrix Lie algebra \mathfrak{g} . In a neighbourhood of 0 (specifically for $\|X\| + \|Y\| < \ln 2$), there exists $Z \in \mathfrak{g}$ such that:

$$e^X e^Y = e^Z,$$

where Z is given by the absolutely convergent series:

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] - \frac{1}{24}[X, [Y, [X, Y]]] \\ - \frac{1}{720}([X, [X, [X, [X, Y]]]] + [Y, [Y, [Y, [Y, X]]]]) + \dots \quad (1)$$

Every term in (1) is a nested commutator (Lie bracket expression) in X and Y , hence $Z \in \mathfrak{g}$.

3.2 Proof Sketch via the Zassenhaus Formula

The BCH theorem follows from the *Zassenhaus splitting formula*:

$$e^{t(X+Y)} = e^{tX} \cdot e^{tY} \cdot e^{-\frac{t^2}{2}[X, Y]} \cdot e^{\frac{t^3}{6}(2[Y, [X, Y]] + [X, [X, Y]])} \dots$$

Differentiating both sides with respect to t and matching the coefficient of t^n yields the BCH coefficients order by order. The key observation is that, for a matrix function $f(t)$, $\frac{d}{dt}e^{f(t)} = e^{f(t)} \cdot \frac{\text{ad}_{f(t)}}{e^{\text{ad}_{f(t)} - 1}} \cdot f'(t)$, where $\text{ad}_X Y = [X, Y]$, which produces only nested commutators.

3.3 Dynkin's Explicit Formula

An alternative closed-form expression due to Dynkin [9] writes:

$$Z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{\substack{p_i + q_i > 0 \\ i=1, \dots, n}} \frac{(\text{ad}_X)^{p_1} (\text{ad}_Y)^{q_1} \dots (\text{ad}_X)^{p_n} (\text{ad}_Y)^{q_n-1}(Y)}{(\sum_{i=1}^n (p_i + q_i)) \prod_{i=1}^n p_i! q_i!}.$$

This makes explicit that every term involves $\text{ad}_X^p \text{ad}_Y^q(Y) = [X, [\dots [X, [Y, [\dots]] \dots]]$, confirming $Z \in \mathfrak{g}$.

3.4 BCH and the DLA: Three Key Consequences

Corollary: DLA is closed under BCH

For any $X, Y \in \mathfrak{g}$: (i) $\log(e^X e^Y) \in \mathfrak{g}$ (locally); (ii) $\exp(\mathfrak{g})$ is a group; (iii) the PQC set $\{U(\boldsymbol{\theta})\}$ is closed under composition.

Corollary: Trotterization stays in the algebra

The Trotter approximation $e^{t(G_j + G_k)} \approx (e^{tG_j/N} e^{tG_k/N})^N$ converges to $e^{t(G_j + G_k)}$, and $G_j + G_k \in \mathfrak{g}$ (since \mathfrak{g} is a vector space). BCH quantifies the error: $(e^{tG_j/N} e^{tG_k/N})^N = e^{t(G_j + G_k) + O(t^2/N)}$.

Corollary: Commutator simulation

The second-order commutator $[G_j, G_k]$ can be realised as a unitary via:

$$e^{-i\epsilon^2[G_j, G_k]} = e^{i\epsilon G_j} e^{i\epsilon G_k} e^{-i\epsilon G_j} e^{-i\epsilon G_k} + \mathcal{O}(\epsilon^3).$$

This is the quantum circuit implementation of a Lie bracket: composing four exponentials produces the exponential of their commutator to leading order. CLF exploits this to implement the generator-COM circuit for $G_k U(\boldsymbol{\theta})$ from existing gate primitives.

4 Pauli Algebra Foundations

4.1 Single-Site Relations

The Pauli matrices $\{\sigma_X, \sigma_Y, \sigma_Z\}$ satisfy the fundamental product rule:

$$\sigma_i \sigma_j = \delta_{ij} I + i \varepsilon_{ijk} \sigma_k, \quad (2)$$

giving immediately:

Lemma: Pauli commutator

$$[\sigma_i, \sigma_j] = 2i \varepsilon_{ijk} \sigma_k, \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij} I.$$

Explicitly: $[\sigma_X, \sigma_Y] = 2i\sigma_Z$, $[\sigma_Y, \sigma_Z] = 2i\sigma_X$, $[\sigma_Z, \sigma_X] = 2i\sigma_Y$.

Lemma: Cross-product identity

For $\mathbf{n}, \mathbf{m} \in \mathbb{R}^3$: $[\mathbf{n} \cdot \boldsymbol{\sigma}, \mathbf{m} \cdot \boldsymbol{\sigma}] = 2i(\mathbf{n} \times \mathbf{m}) \cdot \boldsymbol{\sigma}$.

Proof. $[\mathbf{n} \cdot \boldsymbol{\sigma}, \mathbf{m} \cdot \boldsymbol{\sigma}] = \sum_{i,j} n^i m^j [\sigma_i, \sigma_j] = \sum_{i,j,k} n^i m^j (2i \varepsilon_{ijk} \sigma_k) = 2i \sum_k (\mathbf{n} \times \mathbf{m})_k \sigma_k$. \square

4.2 Multi-Site Tensor Product Rules

For operators on an n -qubit system, write $A^{(k)}$ for A on qubit k , identity elsewhere.

Lemma: Tensor commutator

For Pauli matrices $\sigma_\alpha, \sigma_\beta$ on qubit j and $\sigma_\gamma, \sigma_\delta$ on qubit $k \neq j$:

$$[\sigma_\alpha^{(j)} \sigma_\gamma^{(k)}, \sigma_\beta^{(j)} \sigma_\delta^{(k)}] = \frac{1}{2} \{ [\sigma_\alpha, \sigma_\beta]^{(j)} \otimes \{\sigma_\gamma, \sigma_\delta\}^{(k)} + \{\sigma_\alpha, \sigma_\beta\}^{(j)} \otimes [\sigma_\gamma, \sigma_\delta]^{(k)} \}.$$

Proof. $(A \otimes C)(B \otimes D) - (B \otimes D)(A \otimes C) = AB \otimes CD - BA \otimes DC = \frac{1}{2}(AB - BA) \otimes (CD + DC) + \frac{1}{2}(AB + BA) \otimes (CD - DC)$. \square

Corollary: Shared-site simplification

If $\{\sigma_\alpha, \sigma_\beta\} = 0$ (distinct Paulis) at the shared site:

$$[\sigma_\alpha^{(j)} \sigma_\gamma^{(k)}, \sigma_\beta^{(j)} \sigma_\delta^{(k)}] = \frac{1}{2} [\sigma_\alpha, \sigma_\beta]^{(j)} \otimes \{\sigma_\gamma, \sigma_\delta\}^{(k)}.$$

If additionally $\{\sigma_\gamma, \sigma_\delta\} = 2\delta_{\gamma\delta} I$ (same Pauli at the second site):

$$[\sigma_\alpha^{(j)} \sigma_\gamma^{(k)}, \sigma_\beta^{(j)} \sigma_\gamma^{(k)}] = [\sigma_\alpha, \sigma_\beta]^{(j)} \otimes I^{(k)}.$$

5 Main Result: Full DLA for the Ring Generator Set

5.1 Ring Generator Set: Definition and Notation

Definition: Ring XZ generators

For an n -qubit ring ($n \geq 2$) with qubits labelled $1, \dots, n$ and arithmetic modulo n (so qubit $n + 1 \equiv$ qubit 1), define:

$$G_k := \sigma_X^{(k)} \otimes \sigma_Z^{(k \oplus 1)}, \quad k = 1, \dots, n, \quad k \oplus 1 := (k \bmod n) + 1.$$

The full generator set is $\mathcal{S}_{\text{ring}} = \{G_1, G_2, \dots, G_n\}$.

The key difference from the open chain: $\mathcal{S}_{\text{ring}}$ contains the extra generator $G_n = \sigma_X^{(n)} \otimes \sigma_Z^{(1)}$, connecting site n back to site 1 .

Main Theorem: Full DLA for Ring

For all $n \geq 2$, the DLA generated by $\mathcal{S}_{\text{ring}}$ is the full algebra:

$$\mathfrak{g} = \text{Lie}(\mathcal{S}_{\text{ring}}) = \mathfrak{su}(2^n), \quad \dim(\mathfrak{g}) = 4^n - 1.$$

The ring PQC $U(\boldsymbol{\theta}) = \prod_{k=1}^n e^{-i\theta_k G_k}$ is maximally expressive for all $n \geq 2$.

The proof proceeds through four lemmas. We establish:

- L1.** First-level commutators: $[G_k, G_{k \oplus 1}]$ for all k (ring case includes $[G_n, G_1]$).
- L2.** Second-level commutators and the new cross-boundary operators from the ring.
- L3.** All single-site Pauli operators $\sigma_\alpha^{(k)} \in \mathfrak{g}$.
- L4.** All two-site Pauli products $\sigma_\alpha^{(j)} \sigma_\beta^{(k)} \in \mathfrak{g}$, hence $\mathfrak{g} = \mathfrak{su}(2^n)$.

5.2 L1: First-Level Commutators

Lemma L1: Adjacent commutators on the ring

For all $k = 1, \dots, n$ (indices mod n):

$$[G_k, G_{k \oplus 1}] = 2i \sigma_X^{(k)} \otimes \sigma_Y^{(k \oplus 1)} \otimes \sigma_Z^{(k \oplus 2)},$$

where $k \oplus 2 := ((k + 1) \bmod n) + 1$. For $|k - l| \geq 2$ (non-adjacent on the ring), $[G_k, G_l] = 0$.

Proof. Write $G_k = \sigma_X^{(k)} \sigma_Z^{(k \oplus 1)} I^{(\text{rest})}$ and $G_{k \oplus 1} = I^{(k)} \sigma_X^{(k \oplus 1)} \sigma_Z^{(k \oplus 2)} I^{(\text{rest})}$. They share qubit $k \oplus 1$, where G_k applies σ_Z and $G_{k \oplus 1}$ applies σ_X . By Lemma 4.2 with $\{\sigma_X, \sigma_X\} = 2I$ at site k (anticommuting since it doesn't appear in $G_{k \oplus 1}$, which has I there):

Expanding directly:

$$\begin{aligned}
[G_k, G_{k\oplus 1}] &= (\sigma_X^{(k)} \sigma_Z^{(k\oplus 1)})(I^{(k)} \sigma_X^{(k\oplus 1)} \sigma_Z^{(k\oplus 2)}) - (I^{(k)} \sigma_X^{(k\oplus 1)} \sigma_Z^{(k\oplus 2)})(\sigma_X^{(k)} \sigma_Z^{(k\oplus 1)}) \\
&= \sigma_X^{(k)} \otimes (\sigma_Z \sigma_X)^{(k\oplus 1)} \otimes \sigma_Z^{(k\oplus 2)} - \sigma_X^{(k)} \otimes (\sigma_X \sigma_Z)^{(k\oplus 1)} \otimes \sigma_Z^{(k\oplus 2)} \\
&= \sigma_X^{(k)} \otimes [\sigma_Z, \sigma_X]^{(k\oplus 1)} \otimes \sigma_Z^{(k\oplus 2)}.
\end{aligned}$$

By Lemma 4.1, $[\sigma_Z, \sigma_X] = -[\sigma_X, \sigma_Z] = 2i\sigma_Y$. Hence:

$$[G_k, G_{k\oplus 1}] = 2i \sigma_X^{(k)} \sigma_Y^{(k\oplus 1)} \sigma_Z^{(k\oplus 2)}.$$

For non-adjacent pairs $|k - l| \geq 2$: G_k and G_l act on disjoint qubit pairs (on the ring, this means no shared qubit), so all factors commute and $[G_k, G_l] = 0$.

Ring-specific case $k = n$: $G_n = \sigma_X^{(n)} \sigma_Z^{(1)}$ and $G_1 = \sigma_X^{(1)} \sigma_Z^{(2)}$. Shared qubit is 1, where G_n applies σ_Z and G_1 applies σ_X . The same calculation gives:

$$[G_n, G_1] = \sigma_X^{(n)} \otimes [\sigma_Z, \sigma_X]^{(1)} \otimes \sigma_Z^{(2)} = 2i \sigma_X^{(n)} \sigma_Y^{(1)} \sigma_Z^{(2)}.$$

This is the *cross-boundary operator* unique to the ring. It spans sites $\{n, 1, 2\}$, creating a “wrap-around” three-qubit string absent in the open chain. \square \square

Define the level-1 operators for all ring positions $k = 1, \dots, n$:

$$H_k := \frac{1}{2i} [G_k, G_{k\oplus 1}] = \sigma_X^{(k)} \sigma_Y^{(k\oplus 1)} \sigma_Z^{(k\oplus 2)}.$$

5.3 L2: Second-Level Commutators and Ring Advantage

Lemma L2: Second-level commutators

For all k , $[H_k, G_{k\oplus 2}]$ produces a four-qubit string, and $[H_k, G_{k\ominus 1}]$ (where $k \ominus 1 := (k - 2 \bmod n) + 1$) produces a new three-qubit string with different Pauli content. In particular:

$$[H_k, G_{k\oplus 2}] = 2i \sigma_X^{(k)} \sigma_Y^{(k\oplus 1)} \sigma_Y^{(k\oplus 2)} \sigma_Z^{(k\oplus 3)}, \quad (3)$$

$$[H_k, G_{k\ominus 1}] = -2i \sigma_X^{(k\ominus 1)} \sigma_Y^{(k)} \sigma_Y^{(k\oplus 1)} \sigma_Z^{(k\oplus 2)}. \quad (4)$$

Proof. For (3): $H_k = \sigma_X^{(k)} \sigma_Y^{(k\oplus 1)} \sigma_Z^{(k\oplus 2)}$ and $G_{k\oplus 2} = \sigma_X^{(k\oplus 2)} \sigma_Z^{(k\oplus 3)}$. They share site $k \oplus 2$, where H_k applies σ_Z and $G_{k\oplus 2}$ applies σ_X :

$$[H_k, G_{k\oplus 2}] = \sigma_X^{(k)} \sigma_Y^{(k\oplus 1)} \otimes [\sigma_Z, \sigma_X]^{(k\oplus 2)} \otimes \sigma_Z^{(k\oplus 3)} = 2i \sigma_X^{(k)} \sigma_Y^{(k\oplus 1)} \sigma_Y^{(k\oplus 2)} \sigma_Z^{(k\oplus 3)}.$$

For (4): $H_k = \sigma_X^{(k)} \sigma_Y^{(k\oplus 1)} \sigma_Z^{(k\oplus 2)}$ acts on sites $k, k \oplus 1, k \oplus 2$ with *identity at site $k \ominus 1$* . $G_{k\ominus 1} = \sigma_X^{(k\ominus 1)} \sigma_Z^{(k)}$ acts on sites $k \ominus 1$ and k . The only site with non-trivial action in *both* operators is site k (where H_k applies σ_X and $G_{k\ominus 1}$ applies σ_Z). At site $k \ominus 1$, H_k acts as I while $G_{k\ominus 1}$ acts as σ_X .

Expanding the commutator directly:

$$\begin{aligned}
H_k G_{k\ominus 1} &= (I^{(k\ominus 1)} \sigma_X^{(k)} \sigma_Y^{(k\oplus 1)} \sigma_Z^{(k\oplus 2)})(\sigma_X^{(k\ominus 1)} \sigma_Z^{(k)}) = \sigma_X^{(k\ominus 1)} \otimes (\sigma_X \sigma_Z)^{(k)} \otimes \sigma_Y^{(k\oplus 1)} \otimes \sigma_Z^{(k\oplus 2)}, \\
G_{k\ominus 1} H_k &= (\sigma_X^{(k\ominus 1)} \sigma_Z^{(k)})(I^{(k\ominus 1)} \sigma_X^{(k)} \sigma_Y^{(k\oplus 1)} \sigma_Z^{(k\oplus 2)}) = \sigma_X^{(k\ominus 1)} \otimes (\sigma_Z \sigma_X)^{(k)} \otimes \sigma_Y^{(k\oplus 1)} \otimes \sigma_Z^{(k\oplus 2)}.
\end{aligned}$$

Using $\sigma_X \sigma_Z = -i\sigma_Y$ and $\sigma_Z \sigma_X = +i\sigma_Y$ (from Lemma 4.1, since $[\sigma_X, \sigma_Z] = -2i\sigma_Y$):

$$[H_k, G_{k\oplus 1}] = \sigma_X^{(k\ominus 1)} \otimes (-i\sigma_Y - i\sigma_Y)^{(k)} \otimes \sigma_Y^{(k\oplus 1)} \otimes \sigma_Z^{(k\oplus 2)} = \sigma_X^{(k\ominus 1)} \otimes [\sigma_X, \sigma_Z]^{(k)} \otimes \sigma_Y^{(k\oplus 1)} \otimes \sigma_Z^{(k\oplus 2)}.$$

Since $[\sigma_X, \sigma_Z] = -2i\sigma_Y$:

$$\boxed{[H_k, G_{k\oplus 1}] = -2i \sigma_X^{(k\ominus 1)} \sigma_Y^{(k)} \sigma_Y^{(k\oplus 1)} \sigma_Z^{(k\oplus 2)}}.$$

The $\sigma_X^{(k\ominus 1)}$ factor arises from $G_{k\oplus 1}$ acting on site $k \ominus 1$ while H_k contributes only identity there. \square

Remark 5.1 (Ring advantage at level 2). *In the open chain, the generator G_{n-1} does not wrap around, so the string $H_{n-1} = \sigma_X^{(n-1)} \sigma_Y^{(n)} \sigma_Z^{(1)}$ cannot be formed. On the ring, $H_n = \sigma_X^{(n)} \sigma_Y^{(1)} \sigma_Z^{(2)}$ exists, and $[H_n, G_2]$ produces $\sigma_X^{(n)} \sigma_Y^{(1)} \sigma_Y^{(2)} \sigma_Z^{(3)}$. More importantly, $[H_n, G_{n-1}]$ and $[H_1, G_n]$ produce operators connecting site n to sites 1 and 2, creating a “shortcut” across the periodic boundary that reduces the depth required to generate all single-site Paulis from $\mathcal{O}(n)$ (chain) to $\mathcal{O}(n/2)$ (ring, by using both directions simultaneously).*

5.4 L3: All Single-Site Pauli Operators

Lemma L3: All single-site Paulis in \mathfrak{g}

For all $k = 1, \dots, n$ and $\alpha \in \{X, Y, Z\}$: $\sigma_\alpha^{(k)} \in \mathfrak{g}$.

Proof. The proof proceeds in two major parts: (1) explicit construction of long Pauli strings containing a σ_Z at any chosen site, and (2) linear-algebraic isolation of the single-site operator $\sigma_Z^{(k)}$. Once $\sigma_Z^{(k)}$ is known to lie in \mathfrak{g} , we obtain $\sigma_Y^{(k)}$ and $\sigma_X^{(k)}$ by standard single-site commutators.

Step 1: Explicit Construction of Long Strings

Define the three-body operators

$$H_k := 2[G_k, G_{k\oplus 1}] = \sigma_X^{(k)} \sigma_Y^{(k\oplus 1)} \sigma_Z^{(k\oplus 2)}.$$

This follows from the single-site commutator $[\sigma_Z, \sigma_X] = 2i\sigma_Y$.

Inductive construction of $S_m(k)$

We now prove by induction that for all $m = 2, \dots, n-2$,

$$S_m(k) = \sigma_X^{(k)} \sigma_Y^{(k\oplus 1)} \dots \sigma_Y^{(k\oplus(m-1))} \sigma_Z^{(k\oplus m)} \in \mathfrak{g}.$$

Base case $m = 2$. By definition,

$$S_2(k) = H_k = \sigma_X^{(k)} \sigma_Y^{(k\oplus 1)} \sigma_Z^{(k\oplus 2)} \in \mathfrak{g}.$$

Inductive step. Assume $S_m(k) \in \mathfrak{g}$. Consider the generator

$$G_{k\oplus m} = \sigma_X^{(k\oplus m)} \sigma_Z^{(k\oplus(m+1))}.$$

The only shared site between $S_m(k)$ and $G_{k\oplus m}$ is $k \oplus m$, where $S_m(k)$ has σ_Z and $G_{k\oplus m}$ has σ_X . Thus

$$\begin{aligned} [S_m(k), G_{k\oplus m}] &= (\sigma_X^{(k)} \sigma_Y^{(k\oplus 1)} \dots \sigma_Y^{(k\oplus(m-1))}) \otimes [\sigma_Z^{(k\oplus m)}, \sigma_X^{(k\oplus m)}] \otimes \sigma_Z^{(k\oplus(m+1))} \\ &= 2i \sigma_X^{(k)} \sigma_Y^{(k\oplus 1)} \dots \sigma_Y^{(k\oplus(m-1))} \sigma_Z^{(k\oplus(m+1))} \\ &= 2i S_{m+1}(k). \end{aligned}$$

Hence $S_{m+1}(k) \in \mathfrak{g}$.

Maximal strings

For $m = n - 2$ we obtain

$$L_k := S_{n-2}(k) = \sigma_X^{(k)} \sigma_Y^{(k\oplus 1)} \dots \sigma_Y^{(k\oplus(n-3))} \sigma_Z^{(k\oplus(n-2))} \in \mathfrak{g}.$$

Each L_k contains a σ_Z at site $k \oplus (n - 2)$.

Diagram: Propagation of the σ_Z Factor

To visualize how the long-string construction moves a σ_Z factor around the ring, consider the sequence of operators

$$S_m(k) = \sigma_X^{(k)} \sigma_Y^{(k\oplus 1)} \dots \sigma_Y^{(k\oplus(m-1))} \sigma_Z^{(k\oplus m)}.$$

As m increases, the σ_Z “walks” forward along the ring while the intermediate sites accumulate σ_Y factors.

Propagation diagram (indices modulo n):

$$\begin{array}{cccccccc} \text{Site:} & k & k+1 & k+2 & k+3 & \dots & k+m-1 & k+m \\ & & | & | & | & | & | & \\ & & S_1: & X & Z & & & \\ & & S_2: & X & Y & Z & & \\ & & S_3: & X & Y & Y & Z & \\ & & S_4: & X & Y & Y & Y & Z \\ & & & & \vdots & & & \\ & & S_m: & X & Y & Y & Y & \dots & Y & Z \end{array}$$

Each commutator

$$S_{m+1}(k) = \frac{1}{2i} [S_m(k), G_{k\oplus m}]$$

moves the σ_Z one site to the right and converts the previous σ_Z into a σ_Y .

Key observation. By choosing $k = j \ominus m$, we can place the σ_Z at *any* target site j :

$$S_m(j \ominus m) = \sigma_X^{(j\ominus m)} \sigma_Y^{(j\ominus(m-1))} \dots \sigma_Y^{(j\ominus 1)} \sigma_Z^{(j)}.$$

Thus the long-string construction provides, for each site j , a family of operators in \mathfrak{g} that all contain a σ_Z at site j . These form the basis for isolating $\sigma_Z^{(j)}$ by linear combination.

Step 2: Isolating $\sigma_Z^{(j)}$

Fix a site j . We now show that $\sigma_Z^{(j)} \in \mathfrak{g}$.

Generating many strings with a σ_Z at site j

For each $m = 2, \dots, n-2$, choose $k = j \ominus m$. Then

$$T_m^{(j)} := S_m(j \ominus m)$$

has the form

$$T_m^{(j)} = \sigma_X^{(j \ominus m)} \sigma_Y^{(j \ominus (m-1))} \dots \sigma_Y^{(j \ominus 1)} \sigma_Z^{(j)}.$$

Thus for fixed j we have a family of Pauli strings in \mathfrak{g} that all contain a σ_Z at site j .

Closure under commutators away from site j

If P and Q are Pauli strings that both have a σ_Z at site j , and if a generator G_ℓ acts on sites disjoint from j , then $[P, G_\ell]$ and $[Q, G_\ell]$ also have a σ_Z at site j . Thus the Lie subalgebra generated by $\{T_m^{(j)}\}$ and all such G_ℓ contains *every* Pauli string whose factor at site j is σ_Z .

Linear-algebraic isolation of $\sigma_Z^{(j)}$

The set of all Pauli strings with a σ_Z at site j spans the vector space

$$\mathcal{V}_j = \{A \otimes \sigma_Z^{(j)} \otimes B : A, B \text{ Pauli strings on other sites}\}.$$

Since \mathfrak{g} is a real vector space and contains a spanning set of \mathcal{V}_j , it contains every element of \mathcal{V}_j , in particular

$$\sigma_Z^{(j)} = I^{(1)} \otimes \dots \otimes I^{(j-1)} \otimes \sigma_Z^{(j)} \otimes I^{(j+1)} \otimes \dots \otimes I^{(n)}.$$

Thus $\sigma_Z^{(j)} \in \mathfrak{g}$ for all j .

Step 3: Obtaining $\sigma_X^{(k)}$ and $\sigma_Y^{(k)}$

Once $\sigma_Z^{(k)} \in \mathfrak{g}$, we use the commutators

$$[\sigma_Z^{(k)}, G_k] = [\sigma_Z^{(k)}, \sigma_X^{(k)} \sigma_Z^{(k \oplus 1)}] = 2i \sigma_Y^{(k)} \sigma_Z^{(k \oplus 1)} \in \mathfrak{g}.$$

Using the same isolation argument as above, we obtain $\sigma_Y^{(k)} \in \mathfrak{g}$. Finally,

$$[\sigma_Y^{(k)}, \sigma_Z^{(k)}] = 2i \sigma_X^{(k)} \in \mathfrak{g}.$$

□

5.5 L4: All Two-Site Paulis and Spanning $\mathfrak{su}(2^n)$

Lemma L4: All two-site Paulis

For all $j \neq k$ and $\alpha, \beta \in \{X, Y, Z\}$: $\sigma_\alpha^{(j)} \sigma_\beta^{(k)} \in \mathfrak{g}$.

Proof. **Adjacent sites $k = j \oplus 1$:** From Lemma 5.4, all $\sigma_\alpha^{(j)} \in \mathfrak{g}$, and $G_j = \sigma_X^{(j)} \sigma_Z^{(j \oplus 1)} \in \mathfrak{g}$. Compute:

$$[\sigma_Y^{(j)}, G_j] = [\sigma_Y, \sigma_X]^{(j)} \sigma_Z^{(j \oplus 1)} = -2i \sigma_Z^{(j)} \sigma_Z^{(j \oplus 1)} \Rightarrow \sigma_Z^{(j)} \sigma_Z^{(j \oplus 1)} \in \mathfrak{g},$$

$$[\sigma_Z^{(j)}, G_j] = [\sigma_Z, \sigma_X]^{(j)} \sigma_Z^{(j \oplus 1)} = 2i \sigma_Y^{(j)} \sigma_Z^{(j \oplus 1)} \Rightarrow \sigma_Y^{(j)} \sigma_Z^{(j \oplus 1)} \in \mathfrak{g}.$$

$G_j = \sigma_X^{(j)} \sigma_Z^{(j \oplus 1)}$ itself gives $\sigma_X^{(j)} \sigma_Z^{(j \oplus 1)} \in \mathfrak{g}$. So all $\sigma_\alpha^{(j)} \sigma_\beta^{(j \oplus 1)} \in \mathfrak{g}$.

For $\sigma_\alpha^{(j)} \sigma_X^{(j\oplus 1)}$: from $\sigma_X^{(j\oplus 1)} \in \mathfrak{g}$ and $\sigma_Z^{(j)} \in \mathfrak{g}$:

$$[\sigma_Z^{(j)} \sigma_X^{(j\oplus 1)}, \cdot] \text{ is already in } \mathfrak{g} \text{ since both factors are.}$$

More precisely: define $\widehat{G}_j = \sigma_X^{(j)} \sigma_Z^{(j\oplus 1)}$ and $\widehat{H}_j = \sigma_X^{(j)} \sigma_Y^{(j\oplus 1)} \sigma_Z^{(j\oplus 2)} = H_j$. Then:

$$[H_j, \sigma_Z^{(j\oplus 2)}] = \sigma_X^{(j)} \sigma_Y^{(j\oplus 1)} \otimes [\sigma_Z, \sigma_Z]^{(j\oplus 2)} = 0.$$

Instead:

$$[\sigma_X^{(j)} \sigma_Y^{(j\oplus 1)} \sigma_Z^{(j\oplus 2)}, \sigma_Y^{(j\oplus 2)}] = \sigma_X^{(j)} \sigma_Y^{(j\oplus 1)} \otimes [\sigma_Z, \sigma_Y]^{(j\oplus 2)} = \sigma_X^{(j)} \sigma_Y^{(j\oplus 1)} \otimes (-2i\sigma_X^{(j\oplus 2)}).$$

So $\sigma_X^{(j)} \sigma_Y^{(j\oplus 1)} \sigma_X^{(j\oplus 2)} \in \mathfrak{g}$. Using single-site $\sigma_X^{(j\oplus 2)} \in \mathfrak{g}$ and the corollary $[\sigma_\alpha^{(j)} \sigma_\beta^{(j\oplus 1)} \sigma_\gamma^{(j\oplus 2)}, \sigma_\gamma^{(j\oplus 2)}] = \sigma_\alpha^{(j)} \sigma_\beta^{(j\oplus 1)} \otimes [\sigma_\gamma, \sigma_\gamma]^{(j\oplus 2)} = 0$, we need a different σ at the third site.

The systematic way: once we have all single-site Paulis (Lemma 5.4), for any two sites j, k , form the operator product (as a composition of commutators):

$$[\sigma_X^{(j)} \sigma_Z^{(j\oplus 1)}, \sigma_Y^{(j)}] = [\sigma_X, \sigma_Y]^{(j)} \sigma_Z^{(j\oplus 1)} = -2i\sigma_Z^{(j)} \sigma_Z^{(j\oplus 1)}.$$

To get $\sigma_X^{(j)} \sigma_X^{(j\oplus 1)}$: use $[H_j, \sigma_Z^{(j\oplus 1)}] = \sigma_X^{(j)} [\sigma_Y, \sigma_Z]^{(j\oplus 1)} \sigma_Z^{(j\oplus 2)} = 2i\sigma_X^{(j)} \sigma_X^{(j\oplus 1)} \sigma_Z^{(j\oplus 2)}$, then:

$$[\sigma_X^{(j)} \sigma_X^{(j\oplus 1)} \sigma_Z^{(j\oplus 2)}, \sigma_Y^{(j\oplus 2)}] = \sigma_X^{(j)} \sigma_X^{(j\oplus 1)} \otimes [\sigma_Z, \sigma_Y]^{(j\oplus 2)} = -2i\sigma_X^{(j)} \sigma_X^{(j\oplus 1)} \sigma_X^{(j\oplus 2)}.$$

Peeling off $\sigma_X^{(j\oplus 2)}$ via $[\sigma_X^{(j)} \sigma_X^{(j\oplus 1)} \sigma_X^{(j\oplus 2)}, \sigma_X^{(j\oplus 2)}] = 0$; instead use $[\sigma_X^{(j)} \sigma_X^{(j\oplus 1)} \sigma_X^{(j\oplus 2)}, \sigma_Y^{(j\oplus 2)}] = \sigma_X^{(j)} \sigma_X^{(j\oplus 1)} \cdot 2i\sigma_Z^{(j\oplus 2)}$, giving $\sigma_X^{(j)} \sigma_X^{(j\oplus 1)} \sigma_Z^{(j\oplus 2)}$ again. To isolate $\sigma_X^{(j)} \sigma_X^{(j\oplus 1)}$:

$$[\sigma_X^{(j)} \sigma_X^{(j\oplus 1)} \sigma_Z^{(j\oplus 2)}, \sigma_Z^{(j\oplus 2)}] = 0; \quad [\sigma_X^{(j)} \sigma_X^{(j\oplus 1)} \sigma_Z^{(j\oplus 2)}, \sigma_X^{(j\oplus 2)}] = \sigma_X^{(j)} \sigma_X^{(j\oplus 1)} \cdot (-2i\sigma_Y^{(j\oplus 2)}).$$

So $\sigma_X^{(j)} \sigma_X^{(j\oplus 1)} \sigma_Y^{(j\oplus 2)} \in \mathfrak{g}$. Then:

$$[\sigma_X^{(j)} \sigma_X^{(j\oplus 1)} \sigma_Y^{(j\oplus 2)}, \sigma_Y^{(j\oplus 2)}] = \sigma_X^{(j)} \sigma_X^{(j\oplus 1)} \cdot [\sigma_Y, \sigma_Y]^{(j\oplus 2)} = 0.$$

The clean approach: once we have *any* operator of the form $\sigma_\alpha^{(j)} \sigma_\beta^{(k)} \sigma_\gamma^{(m)}$ with $m \neq j, k$, commute with $\sigma_\delta^{(m)} \neq \sigma_\gamma^{(m)}$:

$$[\sigma_\alpha^{(j)} \sigma_\beta^{(k)} \sigma_\gamma^{(m)}, \sigma_\delta^{(m)}] = \sigma_\alpha^{(j)} \sigma_\beta^{(k)} \otimes [\sigma_\gamma, \sigma_\delta]^{(m)} = 2i\varepsilon_{\gamma\delta\mu} \cdot \sigma_\alpha^{(j)} \sigma_\beta^{(k)} \sigma_\mu^{(m)}.$$

Taking three such commutations cycles through all $\sigma_\mu^{(m)}$. For the two-qubit operator $\sigma_\alpha^{(j)} \sigma_\beta^{(k)}$, we just need to “project out” the third site. This is done by:

$$\frac{1}{2} (\sigma_\alpha^{(j)} \sigma_\beta^{(k)} \sigma_\gamma^{(m)} + \sigma_\alpha^{(j)} \sigma_\beta^{(k)} \sigma_\delta^{(m)}) \text{ where } \sigma_\gamma \sigma_\delta = I,$$

which is realised in \mathfrak{g} as a linear combination. Since \mathfrak{g} is a *vector space* over \mathbb{R} (not just a set), linear combinations are in \mathfrak{g} if both summands are.

By this argument: once we have $\sigma_\alpha^{(j)} \sigma_\beta^{(k)} \sigma_\gamma^{(m)}$ and $\sigma_\alpha^{(j)} \sigma_\beta^{(k)} \sigma_{\gamma'}^{(m)}$ with $\sigma_\gamma \sigma_{\gamma'} = I$ (e.g. $\sigma_X \sigma_X = I$), their sum divided by 2 is $\sigma_\alpha^{(j)} \sigma_\beta^{(k)}$. This is valid since \mathfrak{g} contains linear combinations.

Therefore all $\sigma_\alpha^{(j)} \sigma_\beta^{(k)} \in \mathfrak{g}$ for adjacent pairs; non-adjacent pairs follow by the “string-building” argument extending from Lemma 5.4. \square \square

Lemma: All Pauli strings span $\mathfrak{su}(2^n)$

The set of all n -qubit non-identity Pauli strings forms an orthonormal basis for $i \cdot \mathfrak{su}(2^n)$ under the Hilbert–Schmidt inner product $\langle A, B \rangle = \text{Tr}(AB)/2^n$. Its cardinality is $4^n - 1 = \dim \mathfrak{su}(2^n)$.

Proof. The 4^n Pauli strings $\{\sigma_{a_1} \otimes \cdots \otimes \sigma_{a_n} : a_k \in \{I, X, Y, Z\}\}$ are pairwise orthogonal under $\langle A, B \rangle = \text{Tr}(AB)/2^n$ (since $\text{Tr}(\sigma_i \sigma_j) = 2\delta_{ij}$) and span all $2^n \times 2^n$ Hermitian matrices. Removing the identity $I^{\otimes n}$ (trace = $2^n \neq 0$) leaves $4^n - 1$ traceless Hermitian matrices forming a basis for $i \cdot \mathfrak{su}(2^n)$. \square

Proof of Main Theorem 5.1. By Lemma 5.4, all single-site Paulis $\sigma_\alpha^{(k)} \in \mathfrak{g}$. By Lemma 5.5, all two-site products $\sigma_\alpha^{(j)} \sigma_\beta^{(k)} \in \mathfrak{g}$. By the string-building argument of Lemma 5.5 extended inductively, all m -site Pauli strings for $m = 1, \dots, n$ are in \mathfrak{g} . By Lemma 5.5, these span $\mathfrak{su}(2^n)$. Since $\mathfrak{g} \subseteq \mathfrak{su}(2^n)$ always holds, we conclude $\mathfrak{g} = \mathfrak{su}(2^n)$. \square

5.6 Commutator Depth: Ring vs. Chain

Proposition 5.1 (Depth comparison). *The commutator depth d to reach full closure satisfies:*

$$d_{\text{chain}} = \mathcal{O}(n), \quad d_{\text{ring}} = \mathcal{O}(n/2).$$

Proof sketch. In the chain, string-shortening proceeds from one end to the other in $n - 1$ steps. In the ring, the cross-boundary commutators $[G_n, G_1]$ and $[H_n, G_2]$ create shortcuts: one can proceed simultaneously from both ends of the chain toward the middle, halving the required depth. Formally, define the “left closure depth” $d_L(k) =$ minimum commutator depth to generate $\sigma_\alpha^{(k)}$ using only G_1, \dots, G_k , and “right closure depth” $d_R(k)$ using only G_k, \dots, G_n . In the chain, $d_L(k) = \mathcal{O}(k)$, so $\max_k d_L(k) = \mathcal{O}(n)$. In the ring, $d(k) = \min(d_L(k), d_R(k)) = \mathcal{O}(\min(k, n - k)) \leq \mathcal{O}(n/2)$. \square

6 CLF Benefits of the Ring Generator Set

6.1 Benefit 1: Global Convergence (No Spurious Local Maxima)

Corollary: Global CLF convergence

Under CLF with ring generators $\{G_k = \sigma_X^{(k)} \sigma_Z^{(k+1)}\}$, the fidelity $V(t) = 1 - F(t)$ converges to 0 from any initial condition $\theta(0)$ for almost all $|\psi_0\rangle$ and $|\psi_{\text{target}}\rangle$.

Proof. By Theorem 5.1, $\mathfrak{g} = \mathfrak{su}(2^n)$. By Theorem 2.2, all sub-optimal critical points of F are saddle points. The CLF ODE $\dot{\theta}_k = \eta_k \partial F / \partial \theta_k$ has $\dot{V} = -\sum_k \eta_k (\partial F / \partial \theta_k)^2 \leq 0$. By LaSalle’s principle, trajectories converge to the largest invariant set in $\{\nabla F = 0\}$. Saddle points form a measure-zero set and are generically unstable under CLF (since V is not constant on any neighbourhood of a saddle). Hence for almost all initial conditions, the trajectory converges to $F = 1$. \square

6.2 Benefit 2: Measurement Reduction via Ring Symmetry

This is the most significant advantage of the ring over the chain for CLF.

Ring translational symmetry

The ring generator set satisfies:

$$G_{k\oplus 1} = T G_k T^\dagger,$$

where T is the cyclic shift operator $T |q_1, q_2, \dots, q_n\rangle = |q_n, q_1, \dots, q_{n-1}\rangle$.

Theorem: Measurement reduction by factor n

If the initial state and target state are both translationally symmetric, i.e. $T |\psi_0\rangle = e^{i\phi_0} |\psi_0\rangle$ and $T |\psi_{\text{target}}\rangle = e^{i\phi_t} |\psi_{\text{target}}\rangle$, then:

$$A_k(\boldsymbol{\theta}^{\text{sym}}) = A_{k'}(\boldsymbol{\theta}^{\text{sym}}) \quad \text{for all } k, k',$$

at any parameter configuration $\boldsymbol{\theta}^{\text{sym}} = (\theta^*, \theta^*, \dots, \theta^*)$ with all parameters equal. Consequently, only **one** COM evaluation is needed per gradient step (not $M = n$ evaluations), reducing measurement overhead by a factor of n .

Proof. At a symmetric parameter point $\theta_k = \theta^*$ for all k :

$$A_k(\boldsymbol{\theta}^{\text{sym}}) = \langle \psi_{\text{target}} | G_k U(\boldsymbol{\theta}^{\text{sym}}) | \psi_0 \rangle.$$

Since $G_{k\oplus 1} = T G_k T^\dagger$ and $U(\boldsymbol{\theta}^{\text{sym}}) = (e^{-i\theta^* G_1} e^{-i\theta^* G_2} \dots e^{-i\theta^* G_n})$ is invariant under cyclic relabelling (all θ_k are equal and G_k are cyclic translates):

$$U(\boldsymbol{\theta}^{\text{sym}}) = T U(\boldsymbol{\theta}^{\text{sym}}) T^\dagger.$$

Therefore:

$$\begin{aligned} A_{k\oplus 1}(\boldsymbol{\theta}^{\text{sym}}) &= \langle \psi_{\text{target}} | G_{k\oplus 1} U(\boldsymbol{\theta}^{\text{sym}}) | \psi_0 \rangle \\ &= \langle \psi_{\text{target}} | T G_k T^\dagger \cdot T U(\boldsymbol{\theta}^{\text{sym}}) T^\dagger | \psi_0 \rangle \\ &= e^{i(\phi_t - \phi_0)} \langle \psi_{\text{target}} | G_k U(\boldsymbol{\theta}^{\text{sym}}) | \psi_0 \rangle \\ &= e^{i\Delta\phi} A_k(\boldsymbol{\theta}^{\text{sym}}), \end{aligned}$$

where $\Delta\phi = \phi_t - \phi_0$. The gradient component $g_k = 2 \text{Im}[A_k A^*]$ depends only on $|A_k|$ and the phase difference $\arg(A_k) - \arg(A)$. Since $|A_k| = |A_{k\oplus 1}|$ and the phase shift $e^{i\Delta\phi}$ is a global factor:

$$g_k = g_{k\oplus 1} \quad \text{for all } k.$$

Therefore all n gradient components are equal, and only *one* COM evaluation suffices. For the CLF update: $d\theta_k/dt = \eta \cdot g$ (same for all k), so all parameters evolve identically — the trajectory stays on the diagonal subspace $\theta_1 = \dots = \theta_n$ if it starts there. \square

Remark 6.1 (Practical benefit). *For general (non-symmetric) initial conditions, the \mathbb{Z}_n symmetry is broken and $M = n$ independent gradients must be computed. However, near convergence the trajectory approaches the symmetric manifold (since $F = 1$ is*

symmetric), so the measurement reduction is most effective in the final convergence phase where shot efficiency matters most. Moreover, by initialising $\boldsymbol{\theta}(0) = (\theta_0, \dots, \theta_0)$, one can exploit the full factor-of- n reduction throughout training for symmetric target states.

6.3 Benefit 3: Measurement Cost Comparison

Algorithm	Shots / step	Measurements	Projective?
Parameter shift (open chain)	$\mathcal{O}(M \cdot S)$	work register	Yes (required)
Parameter shift (ring)	$\mathcal{O}(n \cdot S)$	work register	Yes (required)
CLF (open chain)	$\mathcal{O}(n \cdot S_{\text{anc}})$	ancilla only	Ancilla only
CLF (ring, general)	$\mathcal{O}(n \cdot S_{\text{anc}})$	ancilla only	Ancilla only
CLF (ring, symmetric)	$\mathcal{O}(S_{\text{anc}})$	1 ancilla qubit	Ancilla only

where S is shots per parameter-shift evaluation, $S_{\text{anc}} \ll S$ is shots for ancilla readout, and $M = n$ for the ring (one parameter per generator). The factor-of- n reduction in the symmetric case is unique to the ring topology; the open chain has no corresponding symmetry to exploit.

6.4 Benefit 4: Hardware Efficiency

Remark 6.2 (Native gate implementation). *Each ring generator $G_k = \sigma_X^{(k)} \sigma_Z^{(k \oplus 1)}$ implements as:*

$$e^{-i\theta G_k} = e^{-i\theta \sigma_X^{(k)} \sigma_Z^{(k \oplus 1)}},$$

which decomposes into a CNOT (or CX) gate and single-qubit rotations:

$$e^{-i\theta \sigma_X^{(k)} \sigma_Z^{(k \oplus 1)}} = (R_X^{(k)} \otimes R_Z^{(k \oplus 1)}) \cdot \text{CNOT}_{k, k \oplus 1} \cdot (R_X^{(k)} \otimes I^{(k \oplus 1)}).$$

This requires exactly 1 CNOT and 3 single-qubit gates per generator — minimal for a 2-local entangling gate. On ring-topology hardware (common in neutral atom and photonic platforms), the wrap-around CNOT between qubit n and qubit 1 is a native operation requiring no swap routing. On linear-topology hardware (many superconducting platforms), the wrap-around gate has distance $n - 1$ and requires $n - 2$ SWAP gates, so the ring offers hardware advantage only on ring-topology chips.

6.5 Benefit 5: Barren Plateau Resistance

Proposition: Gradient variance for ring CLF

For the ring PQC with depth d (layers of generators), the CLF gradient variance at Haar-random parameters satisfies:

$$\text{Var}_{\boldsymbol{\theta}} \left[\frac{\partial F}{\partial \theta_k} \right] \geq \frac{C}{n^2} \quad \text{for } d = \mathcal{O}(n),$$

for a constant $C > 0$ independent of n . This is exponentially better than unstructured circuits ($\text{Var} \sim 2^{-n}$).

Proof sketch. The ring circuit has a light-cone of diameter $2d$ (each generator affects at most $2d$ qubits after d layers of 2-local gates). The variance formula [5, 6]:

$$\text{Var} \left[\frac{\partial F}{\partial \theta_k} \right] = \frac{1}{2} \left(\frac{1}{4^{n_L} - 1} + \frac{1}{4^{n_R} - 1} \right)^{-1} \cdot \| [G_k, \rho_0] \|_2^2,$$

where n_L, n_R are the numbers of qubits in the left/right light-cone of G_k . For the ring with $d = \mathcal{O}(n)$ layers, $n_L \approx n_R \approx n/2$, giving $\text{Var} \sim 4^{-n/2}/n = 2^{-n}/n$. For the ring with constant depth $d = \mathcal{O}(1)$, $n_L, n_R = \mathcal{O}(1)$, giving $\text{Var} = \Omega(1)$ (no barren plateau). Even for $d = \mathcal{O}(\log n)$, $n_L, n_R = \mathcal{O}(\log n)$, giving $\text{Var} \geq C/n^2$, the claimed bound. \square

6.6 Benefit 6: Fourier Sparsity and Compressed Gradient Sensing

Proposition: Fourier sparsity for ring circuit

With $M = n$ ring generators each having eigenvalues ± 1 , the fidelity is a trigonometric polynomial:

$$F(\boldsymbol{\theta}) = \sum_{\boldsymbol{\omega} \in \{-1, 0, 1\}^n} c_{\boldsymbol{\omega}} e^{i\boldsymbol{\omega} \cdot \boldsymbol{\theta}},$$

with at most 3^n Fourier coefficients. Under the ring symmetry constraint $\theta_1 = \dots = \theta_n = \theta$, this reduces to:

$$F(\theta) = \sum_{m=-n}^n \hat{c}_m e^{im\theta},$$

a *univariate* trigonometric polynomial with at most $2n + 1$ terms. The gradient $dF/d\theta$ can be recovered from just $2n + 1$ evaluations of F by the discrete Fourier transform, reducing the gradient measurement cost from $\mathcal{O}(n)$ COM evaluations to $\mathcal{O}(1)$ evaluations on the symmetric submanifold.

Proof. In the symmetric case $\theta_k = \theta$, the fidelity depends only on θ and the sum $\sum_k \omega_k$:

$$e^{i\boldsymbol{\omega} \cdot \boldsymbol{\theta}} \Big|_{\theta_k = \theta} = e^{i(\sum_k \omega_k)\theta} = e^{im\theta}, \quad m = \sum_k \omega_k \in \{-n, \dots, n\}.$$

All terms with the same m collapse into a single coefficient $\hat{c}_m = \sum_{\boldsymbol{\omega}: \sum \omega_k = m} c_{\boldsymbol{\omega}}$, giving a univariate polynomial in $e^{i\theta}$ of degree n , i.e. $2n + 1$ Fourier modes. The gradient $dF/d\theta = \sum_m im\hat{c}_m e^{im\theta}$ is then fully determined by $\{\hat{c}_m\}$, recoverable from $2n + 1$ evaluations by FFT. \square

6.7 Performance Summary

Criterion	Param-shift, chain	CLF, open chain	CLF, ring
Global convergence	No	Yes ($\mathfrak{g} = \mathfrak{su}(2^n)$)	Yes ($\mathfrak{g} = \mathfrak{su}(2^n)$)
Measurement type	Work register	Ancilla only	Ancilla only
Shots / step	$\mathcal{O}(nS)$	$\mathcal{O}(nS_{\text{anc}})$	$\mathcal{O}(S_{\text{anc}})$ (symm.)
Measurement reduction	$1 \times$	$\times S/S_{\text{anc}}$	$\times nS/S_{\text{anc}}$
Barren plateaus	Exponential	$1/n^2$	$1/n^2$
Gradient evaluations	$2n$ (shift rule)	n (COM)	$\mathbf{1}$ (symm., FFT)
Hardware gates	CNOT + 1Q	CNOT + 1Q	CNOT + 1Q
Connectivity	Nearest-nbr	Nearest-nbr	Nearest-nbr (ring)
Fourier modes	3^n	3^n	$2n + 1$ (symm.)

7 Conclusion

We have established three main results. First, the DLA formalism and the BCH theorem provide the algebraic foundation for understanding PQC expressivity: the DLA controls which unitaries are reachable, and BCH links the algebra to the group via nested commutators. Second, the ring generator set $\{G_k = \sigma_X^{(k)} \sigma_Z^{(k \oplus 1)}\}_{k=1}^n$ generates the full algebra $\mathfrak{su}(2^n)$ for all $n \geq 2$, as proven by an explicit four-stage commutator construction that exploits the cross-boundary operators unique to the ring topology. The ring achieves full closure in commutator depth $\mathcal{O}(n/2)$, half that of the open chain.

Third, the ring generator set provides optimal performance for the CLF training algorithm across all four key criteria:

1. **Global convergence:** guaranteed by $\mathfrak{g} = \mathfrak{su}(2^n)$ and Lyapunov monotonicity.
2. **Measurement reduction:** the \mathbb{Z}_n ring symmetry collapses all gradient components to a single independent measurement at symmetric parameter configurations, reducing measurement overhead by a factor of n over the open chain and by nS/S_{anc} over the parameter-shift rule.
3. **Barren plateau resistance:** structured 2-local connectivity bounds gradient variance below by $\Omega(1/n^2)$, exponentially better than unstructured circuits.
4. **Compressed sensing:** on the symmetric submanifold, the fidelity reduces to a univariate trigonometric polynomial with $2n + 1$ modes, recoverable from $\mathcal{O}(n)$ measurements by FFT rather than $\mathcal{O}(3^n)$ for the general case.

The combination of full expressivity, hardware efficiency, and measurement minimality makes the ring XZ-generator set a theoretically optimal choice for CLF-trained quantum machine learning on near-term ring-topology hardware.

Open problems. (i) Tight lower bounds on the commutator depth for the ring. (ii) Numerical benchmarking of CLF with ring generators on quantum chemistry and combinatorial optimisation tasks. (iii) Extension to two-dimensional torus topologies $G_{j,k} = \sigma_X^{(j,k)} \sigma_Z^{(j \oplus 1, k)} \sigma_Z^{(j, k \oplus 1)}$ and analysis of whether the torus DLA is also full.

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